*On the Factoring of Composite Hypercomplex Number Systems.**

By Heman Burr Leonard.

Introduction.

From the two number systems $E \equiv e_1 \ldots . e_n$ and $F \equiv f_1 \ldots . f_r$, having the multiplication tables $e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3}$ $(i = 1, \ldots, n)$ and $f_{j_1} f_{j_2} = \sum_{j_3} \phi_{j_1 j_2 j_3} f_{j_3}$ $(j = 1, \ldots, r)$, can be formed by multiplication † a number system of nr units $\varepsilon_{i_1 j_1} = e_{i_1} f_{j_1} = f_{j_1} e_{i_1}$, having the multiplication table $\varepsilon_{i_1 j_1} \varepsilon_{i_2 j_2} = (e_{i_1} e_{i_2}) (f_{j_1} f_{j_2}) = \sum_{i_3 j_3} \gamma_{i_1 i_2 i_3} \phi_{j_1 j_2 j_3} \varepsilon_{i_3 j_3}$. In regard to the converse problem Professor Scheffers suggested ‡ in 1891 that there was lacking a serviceable criterion for deciding whether a given system is a compound of systems and also that general theorems concerning the divisors of zero and the characteristic equation were desirable The consideration of these questions has led to the results which are now given in what is to be regarded as a first communication.

Let $A = \sum_i a_i e_i$ and $\bar{A} = \sum_j \bar{a}_j f_j$ be numbers of the systems E and F respectively. Then the number $C = \sum_{ij} a_i \bar{a}_j \varepsilon_{ij}$ *will be called the compound of the numbers* A *and* $\bar{A}$. It is shown in §2 that *if* $\mu_1, \ldots, \mu_n$ *are the roots of the characteristic equation of* A, *and* $\nu_1, \ldots, \nu_r$ *are the roots of the characteristic equation of* $\bar{A}$, *then the roots of the characteristic equation of* C *are* $\mu_i \nu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, r)$.

In §3 is given a method for determining the factor systems of a composite system through the use of the characteristic equation of the composite system.

---

*This paper was read at the meeting of the American Mathematical Society, held at Yale University, September, 1906. An abstract appears in the Bulletin, vol. 13, number 2 (November, 1906), p. 68.

† Scheffers, Mathematische Annalen, vol. 39 (1891), p. 324.

‡ Annalen, vol. 39 (1891), p. 325. "Es fehlt ein brauchbares Criterium dafür, dass ein vorgelegtes System als Product aufgefasst werden kann, und an allgemeinen Sätzen über die Theiler der Null und die charakteristische Gleichung eines solchen Systems."

The method is made clear by its application to the factoring of two composite systems.

A second method, which uses the matrix representation, is given in §4. Because of the difficulty of solving algebraic equations of higher degree than the fourth, this method appears to be the more serviceable one for decomposing composite algebras of the higher orders.

In §5 divisors of zero are considered.

§1.—The Group of the Compound System.

According to Poincaré* and Study† the groups of the algebras E and F are respectively

$$\left.\begin{aligned} G_E : x'_{i_3} &= \sum_{i_1 i_2} \gamma_{i_1 i_2 i_3}\, y_{i_2}\, x_{i_1},\ (i = 1, \ldots, n); \\ G_F : \bar{x}'_{j_3} &= \sum_{j_1 j_2} \phi_{j_1 j_2 j_3}\, \bar{y}_{j_2}\, \bar{x}_{j_1},\ (j = 1, \ldots, r); \end{aligned}\right\} \tag{1}$$

where the x's are variables, the y's parameters. If $X = \sum_{i_1 j_1} x_{i_1 j_1}\, e_{i_1}\, f_{j_1}$, $Y = \sum_{i_2 j_2} y_{i_2 j_2}\, e_{i_2}\, f_{j_2}$, $Z = \sum_{i_3 j_3} z_{i_3 j_3}\, e_{i_3}\, f_{j_3}$, are numbers of the compound algebra $EF \equiv e_i f_j\, (i = 1, \ldots, n;\, j = 1, \ldots, r)$, such that $Z = XY$, then the group of the compound algebra is

$$G_{EF} : z_{i_3 j_3} = \sum_{i_1 i_2 j_1 j_2} \gamma_{i_1 i_2 i_3}\, \phi_{j_1 j_2 j_3}\, y_{i_2 j_2}\, x_{i_1 j_1}\ (i = 1, \ldots, n;\ j = 1, \ldots, r). \tag{2}$$

According to Rados‡ and Burnside,§ the compound $G_E G_F$ of the groups G_E, G_F is obtained as follows: In the function $f = \sum_{i_3 j_3} c'_{i_3 j_3}\, x'_{i_3}\, \bar{x}'_{j_3}$ substitute the values of x'_{i_3} and $\bar{x}'_{j_3}$ and equate the resulting form to $\sum_{i_1 j_1} c_{i_1 j_1}\, x_{i_1}\, \bar{x}_{j_1}$. By comparing coefficients there results

$$c_{i_1 j_1} = \sum_{i_3 i_2 j_3 j_2} \gamma_{i_1 i_2 i_3}\, \phi_{j_1 j_2 j_3}\, y_{i_2}\, \bar{y}_{j_2}\, c'_{i_3 j_3}.\|$$

Therefore the compound of the groups G_E, G_F may be written

$$G_E G_F : c_{i_1 j_1} = \sum_{i_3 i_2 j_3 j_2} \gamma_{i_1 i_2 i_3}\, \phi_{j_1 j_2 j_3}\, y_{i_2 j_2}\, c'_{i_3 j_3}. \tag{3}$$

---

* Poincaré, Comptes Rendus, vol. 99 (1884), pp. 740-742.

† Study, Monatshefte für Math. und Physik, vol. 1 (1890), pp. 283-355.

‡ Rados, Annalen, vol. 48 (1897), pp. 417-424.

§ Burnside, Quarterly Journal of Mathematics, vol. 33 (1902), pp. 80-84.

‖ According to a suggestion derived from a paper by Franklin, this may be called the induced group of G_E and G_F. American Journal of Mathematics, vol. 16 (1894), p. 205.

It can be easily seen that the *transverse** or *converse* or *conjugate* of the group $G_E\, G_F$, designated by $\overbrace{G_E G_F}$, is a subgroup of G_{EF}.

§2.—THE ROOTS OF THE CHARACTERISTIC EQUATION OF THE COMPOUND SYSTEM.

The characteristic equation of E is obtained by writing in the equations of the group G_E $x'_{i_3} = \mu\, y_{i_3}$,† transposing, and since at least one y_i does not vanish, equating the determinant of the coefficients to zero:

$$\begin{vmatrix} \sum\limits_{i_1} x_{i_1}\gamma_{i_1 11} - \mu, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 21} & , \ldots, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 n1} \\ \sum\limits_{i_1} x_{i_1}\gamma_{i_1 12} & , \sum\limits_{i_1} x_{i_1}\gamma_{i_1 22} - \mu, & \ldots, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 n2} \\ \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{i_1} x_{i_1}\gamma_{i_1 1n} & , \sum\limits_{i_1} x_{i_1}\gamma_{i_1 2n} & , \ldots, & \sum\limits_{i_1} x_{i_1}\gamma_{i_1 nn} - \mu \end{vmatrix} = 0. \qquad (4)$$

From another point of view the scalar μ must satisfy the equation (4) in order that for the general number x of E there should exist a number y such that $xy = \mu y$.‡

Similarly, if one writes $\nu \bar{y}_{j_3}$ for $\bar{x}'_{j_3}$ in the equations of the group G_F and transposes, the determinant of the coefficients of the y's

$$\begin{vmatrix} \sum\limits_{j_1} \bar{x}_{j_1}\phi_{j_1 j_2 j_3} - \nu\delta_{j_2 j_3}\text{§} \\ j_2,\, j_3 = 1, \ldots, r \end{vmatrix} = 0, \qquad (5)$$

expresses the fact that there exists a $y \neq 0$, such that $\bar{x}\bar{y} = \nu\bar{y}$.

The characteristic equation of the compound system EF‖ obtained in the same manner from the group G_{EF} is:

---

* In the American Journal of Mathematics, vol. 12 (1890), p. 340, Taber attributes the term transverse to Cayley, the term converse to Charles Peirce, and the term conjugate to Hamilton.

† Scheffers, Annalen, vol. 39 (1891), p. 303.

‡ If we let $x'_{i_3} = \mu x_{i_3}$, an equation similar to (4) is obtained, which expresses the fact that a number y exists such that $yx = \mu y$. In the present investigation, we follow Cartan (Annales de la Faculté des Sciences de Toulouse, vol. 12 (1898), p. B 17) in restricting our attention to the equation (4).

§ Here and hereafter in this paper $\delta_{j_2 j_3} = \begin{cases} 1, \text{ for } j_2 = j_3 \\ 0, \text{ for } j_2 \neq j_3 \end{cases}$ according to the Kronecker usage.

‖ At first glance one might surmise that the characteristic equation of the compound system EF should be

$$\begin{vmatrix} \sum\limits_{i_1} x_{i_1}\gamma_{i_1 i_2 i_3} - \mu\delta_{i_2 i_3} \\ i_2,\, i_3 = 1, \ldots, n \end{vmatrix} \cdot \begin{vmatrix} \sum\limits_{j_1} \bar{x}_{j_1}\phi_{j_1 j_2 j_3} - \mu\delta_{j_2 j_3} \\ j_2,\, j_3 = 1, \ldots, r \end{vmatrix} = 0,$$

which is in fact the characteristic equation of the reducible system having E and F for its constituents. Annalen, vol. 39 (1891), p. 320.

$$\begin{vmatrix} \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 11}-\zeta, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 21}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 r1}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 21}\Phi_{j_1 11}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 n1}\Phi_{j_1 r1} \\ \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 12}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 22}-\zeta, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 r2}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 21}\Phi_{j_1 12}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 n1}\Phi_{j_1 r2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 1r}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 2r}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 11}\Phi_{j_1 rr}-\zeta, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 21}\Phi_{j_1 1r}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 n1}\Phi_{j_1 rr} \\ \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 12}\Phi_{j_1 11}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 12}\Phi_{j_1 21}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 12}\Phi_{j_1 r1}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 22}\Phi_{j_1 11}-\zeta, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 n2}\Phi_{j_1 r1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 1n}\Phi_{j_1 1r}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 1n}\Phi_{j_1 2r}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 1n}\Phi_{j_1 rr}, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 2n}\Phi_{j_1 1r}, & \ldots, & \sum\limits_{i_1 j_1} x_{i_1 j_1}\gamma_{i_1 nn}\Phi_{j_1 rr}-\zeta \end{vmatrix} = 0. \qquad (6)$$

The characteristic equation of the special number $A = a_i e_i$ of E is obtained from (4) by writing $a_{i_1} = x_{i_1}$. Similarly the characteristic equation of $\bar{A} = \bar{a}_j f_j$ is obtained from (5) by writing $\bar{a}_{j_1} = \bar{x}_{j_1}$. Since the characteristic equation of a matrix is the same as that of its conjugate, the characteristic equation of the compound C of these numbers (Introduction) is obtained by writing $a_{i_1} \bar{a}_{j_1} = x_{i_1 j_1}$ in (6). We proceed to show that if the roots of

$$\left| \begin{array}{l} \sum\limits_{i_1} a_{i_1} \gamma_{i_1 i_2 i_3} - \mu \delta_{i_2 i_3} \\ i_2, i_3 = 1, \ldots, n \end{array} \right| = 0 \tag{4'}$$

are $\mu_1, \ldots, \mu_n$ and those of

$$\left| \begin{array}{l} \sum\limits_{j_1} \bar{a}_{j_1} \phi_{j_1 j_2 j_3} - \nu \delta_{j_2 j_3} \\ j_2, j_3 = 1, \ldots, r \end{array} \right| = 0 \tag{5'}$$

are $\nu_1, \ldots, \nu_r$, then the nr roots of the characteristic equation of the compound number C

$$\left| \begin{array}{l} \sum\limits_{i_1 j_1} a_{i_1} \bar{a}_{j_1} \gamma_{i_1 i_2 i_3} \phi_{j_1 j_2 j_3} - \zeta \delta_{i_2 i_3} \delta_{j_2 j_3} \\ j_2 = 1, \ldots, r; \text{ then } i_2 = 1, \ldots, n \\ j_3 = 1, \ldots, r; \text{ then } i_3 = 1, \ldots, n \end{array} \right| = 0 \tag{6'}$$

are $\mu_i \nu_j$ $(i = 1, \ldots, n;\ j = 1, \ldots, r)$.

If $\mu_1, \ldots, \mu_n$ are the roots of the equation (4′), there are n linear functions $L_1, \ldots, L_n$, which are transformed by any particular substitution S_E of the group G_E into $\mu_1 L_1, \ldots, \mu_n L_n$. Likewise if $\nu_1, \ldots, \nu_r$ are the roots of (5′), there exist r linear functions $\bar{L}_1, \ldots, \bar{L}_r$, which are transformed by any partic ular substitution S_F of the group G_F into $\nu_1 \bar{L}_1, \ldots, \nu_r \bar{L}_r$. Evidently the functions $L_i \bar{L}_j$ are transformed by the successive operation S_E, S_F into $\mu_i \nu_j L_i \bar{L}_j$ The same result is obtained by transforming $L_i \bar{L}_j$ by $\smile{S_E S_F}$. But $\smile{S_E S_F}\,(L_i \bar{L}_j) = \zeta_{ij} L_i \bar{L}_j$.* Therefore $\zeta_{ij} = \mu_i \nu_j$ and the theorem is proved.

§3.—Factoring of Composite Systems by Characteristic Equation Method.

I. The multiplication tables of the systems E and F being given, the multiplication table of the compound system EF is determined by the consideration that its nr units are $e_i f_j$. If the characteristic equations of a number A of E and

* Franklin, American Journal of Mathematics, vol. 16 (1894), p. 205.

a number $\bar{A}$ of F are given, the characteristic equation of the compound number C can be determined. Let

$$\mu^n - p_1\mu^{n-1} + p_2\mu^{n-2} - \ldots\ldots + (-1)^n p_n = 0 \qquad (4'')$$

and

$$\nu^r - q_1\nu^{r-1} + q_2\nu^{r-2} - \ldots\ldots + (-1)^r q_r = 0 \qquad (5'')$$

be the characteristic equations of A and $\bar{A}$ and let

$$\zeta^{nr} - s_1\zeta^{nr-1} + s_2\zeta^{nr-2} - \ldots\ldots + (-1)^{nr} s_{nr} = 0 \qquad (6'')$$

be the characteristic equation of the compound number C. The coefficients s can be determined in terms of p and q. Since the roots of $(6'')$ are $\mu_i\nu_j$, the coefficients s of $(6'')$ are calculated in terms of p and q by means of the symmetric functions of the roots of $(4'')$ and $(5'')$. The converse problem is considered from two points of view. In §4 from a given compound system are derived the factor systems. In this section (§3) from the characteristic equation of a general number C of the compound system are calculated the characteristic equations of corresponding general numbers A and $\bar{A}$ of the factor systems.

That the problems of §4 and §3 are not strictly identical can best be made clear by an illustration. The characteristic equation of a general number of the system

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_1	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_2	h_2	0	h_4	0	h_6	0	h_8	0
h_3	h_3	h_4	h_7	h_8	0	0	0	0
h_4	h_4	0	h_8	0	0	0	0	0
h_5	h_5	h_6	0	0	0	0	0	0
h_6	h_6	0	0	0	0	0	0	0
h_7	h_7	h_8	0	0	0	0	0	0
h_8	h_8	0	0	0	0	0	0	0

is $(x_1 - \zeta)^8 = 0$. By the methods explained later in this section, the characteristic equations of general numbers of the factor systems are calculated to be $\mu^2 - p_1\mu + \frac{p_1^2}{4} = 0$ and $\nu^4 - q_1\nu^3 + \frac{3}{8}q_1^2\nu^2 - \frac{1}{16}q_1^3\nu + \frac{1}{256}q_1^4 = 0$. The first

one is evidently the characteristic equation of a general number of the Cayley two-unit system.* On the other hand the system belonging to $\left(\nu - \frac{q_1}{4}\right)^4 = 0$ is not uniquely determined, since all of the following systems have the same equation: †

IV_1.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_3	e_4	0
e_3	e_3	e_4	0	0
e_4	e_4	0	0	0

IV_3.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	λe_4	e_4	0
e_3	e_3	$-e_4$	e_4	0
e_4	e_4	0	0	0

IV_4.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	0	0
e_3	e_3	0	e_4	0
e_4	e_4	0	0	0

IV_5.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_4	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

IV_8.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	0	e_4	0
e_3	e_3	$-e_4$	0	0
e_4	e_4	0	0	0

IV_9.

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	0	0	0
e_3	e_3	0	0	0
e_4	e_4	0	0	0

However by the method of §4 the factor systems are found to be

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

and

	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_4	0	0
f_3	f_3	0	0	0
f_4	f_4	0	0	0.

* $e^2_1 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e^2_2 = 0$.

† Scheffers, Annalen, vol. 39 (1891), p. 352. The characteristic equations there given are in the reduced form.

Nevertheless in some respects the method of this section is more powerful than that of §4. Thus the system

$$\begin{array}{c|cccc} & h_1 & h_2 & h_3 & h_4 \\ \hline h_1 & h_1 & h_2 & h_3 & h_4 \\ h_2 & h_2 & 0 & h_4 - h_3 & h_4 - h_3 \\ h_3 & h_3 & h_4 - h_3 & -h_1 & -h_1 - h_2 \\ h_4 & h_4 & h_4 - h_3 & -h_1 - h_2 & -h_1 - 2h_2 \end{array}$$

can not be resolved by the method of §4; but the characteristic equation of its general number is $\zeta^4 - 4x_1\zeta^3 + \zeta^2(6x_1^2 + 2x_3^2 + 4x_3x_4 + 2x_4^2) - \zeta(4x_1^3 + 4x_1x_3^2 + 8x_1x_3x_4 + 4x_1x_4^2) + (x_1^4 + 2x_1^2x_3^2 + 4x_1^2x_3x_4 + 2x_1^2x_4^2 + x_3^4 + 4x_3^3x_4 + 6x_3^2x_4^2 + 4x_3x_4^3 + x_4^4) = 0$ and by the method of this section the characteristic equations of general numbers of its factor systems are found to be $\mu^2 - p_1\mu + \frac{p_1^2}{4} = 0$ and $\nu^2 - q_1\nu + \frac{q_1^2}{4x_1^2}(x_1^2 + x_3^2 + x_4^2 + 2x_3x_4) = 0$. The factor systems* belong to the types

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_1 & e_2 \\ e_2 & e_2 & 0 \end{array} \quad \text{and} \quad \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & f_1 & f_2 \\ f_2 & f_2 & -f_1. \end{array}$$

II. We start with the simplest composite systems, namely those of order four, whose factors must be two two-unit systems. Assume as the characteristic equation of a general number of the compound system $\zeta^4 - s_1\zeta^3 + s_2\zeta^2 - s_3\zeta + s_4 = 0$. For the characteristic equations of general numbers of the two factor systems may be assumed $\mu^2 - p_1\mu + p_2 = 0$ and $\nu^2 - q_1\nu + q_2 = 0$. By forming the symmetric functions of the roots of these equations, the following relations are obtained:

$$\left.\begin{aligned} s_1 &= p_1 q_1 \\ s_2 &= p_1^2 q_2 - 2p_2 q_2 + p_2 q_1^2 \\ s_3 &= p_1 p_2 q_1 q_2 \\ s_4 &= p_2^2 q_2^2. \end{aligned}\right\} \qquad (7)$$

* When these two systems are compounded and the following linear transformations are made on the units, $h_4 = g_3 + g_4$, $h_1 = g_1$, $h_2 = g_2$, $h_3 = g_3$, the given form of the composite system results.

An obvious condition on the s's is $s_1^2 s_4 = s_3^2$. The formation of the characteristic equation for a general number of the given system furnishes the values for the s's. From the above relations p_2 can be determined in terms of p_1 and q_2 in terms of q_1. Thus the nature of the roots of the characteristic equations of general numbers of the two factor systems is determined.

For example, consider the system

$$\begin{array}{c|cccc} & h_1 & h_2 & h_3 & h_4 \\ \hline h_1 & h_1 & h_2 & h_3 & h_4 \\ h_2 & h_2 & 0 & h_4 & 0 \\ h_3 & h_3 & h_4 & -h_1 & -h_2 \\ h_4 & h_4 & 0 & -h_2 & 0. \end{array}$$

The characteristic equation of a general number of the system is

$$\begin{vmatrix} x_1-\zeta, & 0\;, & -x_3, & 0 \\ x_2\;, & x_1-\zeta, & -x_4, & -x_3 \\ x_3\;, & 0\;, & x_1-\zeta, & 0 \\ x_4\;, & x_3\;, & x_2\;, & x_1-\zeta \end{vmatrix} = 0,$$

which, multiplied out, is

$$\zeta^4 - \zeta^3(4x_1) + \zeta^2(6x_1^2 + 2x_3^2) - \zeta(4x_1^3 + 4x_1x_3^2) + (x_1^4 + 2x_1^2x_3^2 + x_3^4) = 0.$$

Substituting in the above relations (7)

$$\begin{aligned} p_1q_1 &= 4x_1 \\ p_1^2q_2 - 2p_2q_2 + p_2q_1^2 &= 6x_1^2 + 2x_3^2. \\ p_1p_2q_1q_2 &= 4x_1(x_1^2 + x_3^2) \\ p_2^2q_2^2 &= x_1^4 + 2x_1^2x_3^2 + x_3^4 = (x_1^2 + x_3^2)^2. \end{aligned}$$

Combining and solving, the following values for the coefficients are obtained:

$$p_2 = \frac{p_1^2}{4} \quad \text{or} \quad \frac{p_1^2(x_1^2 + x_3^2)}{4x_1^2}$$

$$q_2 = \frac{q_1^2(x_1^2 + x_3^2)}{4x_1^2} \quad \text{or} \quad \frac{q_1^2}{4}.$$

Substituting the first set of these values, the characteristic equations of general numbers of the factor systems become $\mu^2 - p_1\mu + \frac{p_1^2}{4} = 0$ and $\nu^2 - q_1\nu + \frac{q_1^2(x_1^2 + x_3^2)}{4x_1^2} = 0$. The first has equal roots and indicates the Cayley system. The second has complex roots and indicates the ordinary complex system.

The substitution of the second set of these values gives the same equations in reverse order.

III. The second lowest composite number is six and a compound system of six units must have for its factor systems a two-unit and a three-unit system. Assume as the characteristic equation of a general number of the composite system

$$\zeta^6 - s_1\zeta^5 + s_2\zeta^4 - s_3\zeta^3 + s_4\zeta^2 - s_5\zeta + s_6 = 0$$

and for general numbers of the two factor systems $\mu^3 - p_1\mu^2 + p_2\mu - p_3 = 0$ and $\nu^2 - q_1\nu + q_2 = 0$. By forming the symmetric functions of the roots of these equations, the following relations are obtained:

$$\left.\begin{aligned}
s_1 &= p_1 q_1 \\
s_2 &= p_1^2 q_2 - 2p_2 q_2 + p_2 q_1^2 \\
s_3 &= p_1 p_2 q_1 q_2 + p_3 q_1^3 - 3p_3 q_1 q_2 \\
s_4 &= p_2^2 q_2^2 + p_1 p_3 q_1^2 q_2 - 2p_1 p_3 q_2^2 \\
s_5 &= p_2 p_3 q_1 q_2^2 \\
s_6 &= p_3^2 q_2^3.
\end{aligned}\right\} \tag{8}$$

The formation of the characteristic equation for a general number of the given system furnishes the values for the s's. From the above relations p_2 and p_3 can be determined in terms of p_1, and q_2 in terms of q_1. This enables one to decide the nature of the roots of the characteristic equations of general numbers of the two factor systems.*

* The above six equations contain five unknowns p_1, p_2, p_3, q_1, q_2, the elimination of which gives certain syzygies among the s's. When these relations are fulfilled, the number (whose characteristic equation is being considered) is a compound. The eliminations of the p's and q's are too lengthy to be taken up at present.

For example, consider the system

	h_1	h_2	h_3	h_4	h_5	h_6
h_1	h_1	h_2	h_3	h_4	h_5	h_6
h_2	h_2	0	h_4	0	h_6	0
h_3	h_3	h_4	0	0	0	0
h_4	h_4	0	0	0	0	0
h_5	h_5	h_6	0	0	0	0
h_6	h_6	0	0	0	0	0.

The characteristic equation of a general number of this system is

$$\zeta^6 - 6x_1\zeta^5 + 15x_1^2\zeta^4 - 20x_1^3\zeta^3 + 15x_1^4\zeta^2 - 6x_1^5\zeta + x_1^6 = 0.$$

Substituting in the above relations (8)

$$\begin{aligned} p_1 q_1 &= 6x_1 \\ p_1^2 q_2 - 2p_2 q_2 + p_2 q_1^2 &= 15x_1^2 \\ p_1 p_2 q_1 q_2 + p_3 q_1^3 - 3p_3 q_1 q_2 &= 20x_1^3 \\ p_2^2 q_2^2 + p_1 p_3 q_1^2 q_2 - 2p_1 p_3 q_2^2 &= 15x_1^4 \\ p_2 p_3 q_1 q_2^2 &= 6x_1^5 \\ p_3^2 q_2^3 &= x_1^6. \end{aligned}$$

Combining and solving, from the first, second, fifth, and sixth of these relations the following values for the coefficients are obtained:

$$p_1 = -4p_3^{\frac{1}{3}},\ 3p_3^{\frac{1}{3}},\ \text{or}\ 3p_3^{\frac{1}{3}}.$$

With the first of these values are associated

$$\begin{aligned} p_2 &= -\tfrac{1}{4} p_1^2 \\ p_3 &= -\tfrac{1}{64} p_1^3 \\ q_1 &= \frac{6x_1}{p_1} \\ q_2 &= \frac{16x_1^2}{p_1^2} = \frac{4q_1^2}{9} \end{aligned}$$

With the second of these values are associated

$$\begin{aligned} p_2 &= \tfrac{1}{3} p_1^2 \\ p_3 &= \tfrac{1}{27} p_1^3 \\ q_1 &= \frac{6x_1}{p_1} \\ q_2 &= \frac{9x_1^2}{p_1^2} = \tfrac{1}{4} q_1^2 . \end{aligned}$$

In calculating these values the third relation and the fourth relation were not used and we find that the third relation is not satisfied by the first set of values. The other set of values satisfies all six relations. The characteristic equations of general numbers of the factor systems are therefore

$$\nu^2 - q_1\nu + \tfrac{1}{4}q_1^2 = 0, \qquad \nu = \tfrac{1}{2}q_1, \qquad \tfrac{1}{2}q_1,$$

and

$$\mu^3 - p_1\mu^2 + \tfrac{1}{3}p_1^2\mu - \tfrac{1}{27}p_1^3 = 0, \qquad \mu = \tfrac{1}{3}p_1, \qquad \tfrac{1}{3}p_1, \qquad \tfrac{1}{3}p_1.$$

The first has equal roots and indicates the Cayley two-unit system. The second has three equal roots and indicates one of the two types of systems *

$$\mathrm{III}_1. \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_2 & e_3 \\ e_2 & e_2 & e_3 & 0 \\ e_3 & e_3 & 0 & 0 \end{array} \qquad \text{and} \qquad \mathrm{III}_3. \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & e_2 & e_3 \\ e_2 & e_2 & 0 & 0 \\ e_3 & e_3 & 0 & 0. \end{array}$$

Trial shows that the second system is the desired three-unit factor system.

IV. The treatment of composite systems of higher orders has been made along similar lines, but it is not considered advisable to take space at this time to give the details.

§4.—FACTORING OF COMPOSITE SYSTEMS BY MATRIX METHOD.

I. "The relative form" of an associative algebra, developed by Charles S. Peirce,† is really a representation of the algebra in matrix form.‡ Given the multiplication table of an algebra in the form

$$\left.\begin{array}{l} e_1^2 \;\; = \gamma_{111}e_1 + \gamma_{112}e_2 + \gamma_{113}e_3 + \cdots + \gamma_{11n}e_n \\ e_1e_2 = \gamma_{121}e_1 + \gamma_{122}e_2 + \gamma_{123}e_3 + \cdots + \gamma_{12n}e_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_1e_n = \gamma_{1n1}e_1 + \gamma_{1n2}e_2 + \gamma_{1n3}e_3 + \cdots + \gamma_{1nn}e_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_n^2 \;\; = \gamma_{nn1}e_1 + \gamma_{nn2}e_2 + \gamma_{nn3}e_3 + \cdots + \gamma_{nnn}e_n, \end{array}\right\} \qquad (9)$$

* Scheffers, Annalen, vol. 39 (1891), p. 353.

† Peirce, American Journal of Mathematics, vol. 4 (1882), p. 221. Proceedings of the American Academy of Arts and Sciences, May 11, 1875, whole series vol. 10, p. 392. Also Benjamin Peirce, same vol., p. 397.

‡ Shaw, Transactions of the American Mathematical Society, vol. 4 (1903), p. 252.

the matrix representation (the ideal units $I:A$, $J:A$, etc., excluded) is in double suffix notation*

$$\left.\begin{aligned}
g_1' = {} & \gamma_{111} g_{11} + \gamma_{112} g_{21} + \gamma_{113} g_{31} + \cdots + \gamma_{11n} g_{n1} \\
& + \gamma_{121} g_{12} + \gamma_{122} g_{22} + \gamma_{123} g_{32} + \cdots + \gamma_{12n} g_{n2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \gamma_{1n1} g_{1n} + \gamma_{1n2} g_{2n} + \gamma_{1n3} g_{3n} + \cdots + \gamma_{1nn} g_{nn} \\
g_2' = {} & \gamma_{211} g_{11} + \gamma_{212} g_{21} + \gamma_{213} g_{31} + \cdots + \gamma_{21n} g_{n1} \\
& + \gamma_{221} g_{12} + \gamma_{222} g_{22} + \gamma_{223} g_{32} + \cdots + \gamma_{22n} g_{n2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \gamma_{2n1} g_{1n} + \gamma_{2n2} g_{2n} + \gamma_{2n3} g_{3n} + \cdots + \gamma_{2nn} g_{nn} \\
g_i' = {} & \gamma_{i11} g_{11} + \gamma_{i12} g_{21} + \gamma_{i13} g_{31} + \cdots + \gamma_{i1n} g_{n1} \\
& + \gamma_{i21} g_{12} + \gamma_{i22} g_{22} + \gamma_{i23} g_{32} + \cdots + \gamma_{i2n} g_{n2} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \gamma_{in1} g_{1n} + \gamma_{in2} g_{2n} + \gamma_{in3} g_{3n} + \cdots + \gamma_{inn} g_{nn}.
\end{aligned}\right\} \qquad (10)$$

Taber proves that matrices of composite order can be factored.† From this a suggestion comes for factoring a composite algebra. Put the composite algebra into matrix form, factor the matrices, and translate the factors back into number systems.

On account of the difficulty of describing this method in words, it is placed before the reader in the solution of four examples. These illustrations are sufficient to make evident the scheme, which is perfectly general.

II. The first system to be considered is the one whose multiplication table is

	h_1	h_2	h_3	h_4
h_1	h_1	h_2	h_3	h_4
h_2	h_2	0	h_4	0
h_3	h_3	h_4	$-h_1$	$-h_2$
h_4	h_4	0	$-h_2$	0

* Study, Encyklopaedie der Math. Wissen., vol. 1, p. 170.

† Taber, American Journal of Mathematics, vol. 12 (1890), p. 391.

If this is a composite system, its factors must be two two-unit systems. Assume for them

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_{11} & e_{12} \\ e_2 & e_{21} & e_{22} \end{array} \quad \text{and} \quad \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22}. \end{array}$$

Symbolically the compound system is

$$\begin{array}{c|cccc} & e_1 f_1 & e_2 f_1 & e_1 f_2 & e_2 f_2 \\ \hline e_1 f_1 & e_{11} f_{11} = g_{11} & e_{12} f_{11} = g_{12} & e_{11} f_{12} = g_{13} & e_{12} f_{12} = g_{14} \\ e_2 f_1 & e_{21} f_{11} = g_{21} & e_{22} f_{11} = g_{22} & e_{21} f_{12} = g_{23} & e_{22} f_{12} = g_{24} \\ e_1 f_2 & e_{11} f_{21} = g_{31} & e_{12} f_{21} = g_{32} & e_{11} f_{22} = g_{33} & e_{12} f_{22} = g_{34} \\ e_2 f_2 & e_{21} f_{21} = g_{41} & e_{22} f_{21} = g_{42} & e_{21} f_{22} = g_{43} & e_{22} f_{22} = g_{44}. \end{array} \tag{11}$$

Substituting in the above formulas (10), the following expressions result

$$\begin{aligned} g_1' = & \ (1) g_{11} + 0\, g_{21} + 0\, g_{31} + 0\, g_{41} \\ & + 0\, g_{12} + (1) g_{22} + 0\, g_{32} + 0\, g_{42} \\ & + 0\, g_{13} + 0\, g_{23} + (1) g_{33} + 0\, g_{43} \\ & + 0\, g_{14} + 0\, g_{24} + 0\, g_{34} + (1) g_{44} \\ = & \ g_{11} + g_{22} + g_{33} + g_{44}, \end{aligned}$$

$$\begin{aligned} g_2' = & \ 0\, g_{11} + (1) g_{21} + 0\, g_{31} + 0\, g_{41} \\ & + 0\, g_{12} + 0\, g_{22} + 0\, g_{32} + 0\, g_{42} \\ & + 0\, g_{13} + 0\, g_{23} + 0\, g_{33} + (1) g_{43} \\ & + 0\, g_{14} + 0\, g_{24} + 0\, g_{34} + 0\, g_{44} \\ = & \ g_{21} + g_{43}, \end{aligned}$$

$$\begin{aligned} g_3' = & \ 0\, g_{11} + 0\, g_{21} + (1) g_{31} + 0\, g_{41} \\ & + 0\, g_{12} + 0\, g_{22} + 0\, g_{32} + (1) g_{42} \\ & + (-1) g_{13} + 0\, g_{23} + 0\, g_{33} + 0\, g_{43} \\ & + 0\, g_{14} + (-1) g_{24} + 0\, g_{34} + 0\, g_{44} \\ = & \ g_{31} + g_{42} - g_{13} - g_{24}, \end{aligned}$$

$$\begin{aligned} g_4' = & \ 0\, g_{11} + 0\, g_{21} + 0\, g_{31} + (1) g_{41} \\ & + 0\, g_{12} + 0\, g_{22} + 0\, g_{32} + 0\, g_{42} \\ & + 0\, g_{13} + (-1) g_{23} + 0\, g_{33} + 0\, g_{43} \\ & + 0\, g_{14} + 0\, g_{24} + 0\, g_{34} + 0\, g_{44} \\ = & \ g_{41} - g_{23}. \end{aligned}$$

Substituting the symbolic products from (11) and factoring each expression:

$$\begin{aligned}
g_1' &= e_{11}f_{11} + e_{22}f_{11} + e_{11}f_{22} + e_{22}f_{22} = (e_{11} + e_{22})\,(f_{11} + f_{22}) \\
g_2' &= e_{21}f_{11} + e_{21}f_{22} \qquad\qquad\qquad\; = \qquad e_{21}\;(f_{11} + f_{22}) \\
g_3' &= e_{11}f_{21} + e_{22}f_{21} - e_{11}f_{12} - e_{22}f_{12} = (e_{11} + e_{22})\,(f_{21} - f_{12}) \\
g_4' &= e_{21}f_{21} - e_{21}f_{12} \qquad\qquad\qquad\; = \qquad e_{21}\;(f_{21} - f_{12}).
\end{aligned}$$

The units of one system are represented by $e_{11} + e_{22}$ and e_{21}. The units of the other system are represented by $f_{11} + f_{22}$ and $f_{21} - f_{12}$. The law for the combination of such expressions is $g_{rs}\, g_{qt} = g_{rt}\, \delta_{sq}$. Multiplying out according to this law, we get for the first factor system

$$\begin{array}{c|cc}
 & e_{11} + e_{22} & e_{21} \\
\hline
e_{11} & e_{11} + 0 & 0 \\
+\; e_{22} & +\,0 + e_{22} & +\,e_{21} \\
e_{21} & e_{21} + 0 & 0.
\end{array}$$

In ordinary notation this system is

$$\begin{array}{c|cc}
 & e_1 & e_2 \\
\hline
e_1 & e_1 & e_2 \\
e_2 & e_2 & 0.
\end{array}$$

For the second factor system, we obtain

$$\begin{array}{c|cc}
 & f_{11} + f_{22} & f_{21} - f_{12} \\
\hline
f_{11} & f_{11} + 0 & 0 - f_{12} \\
+\; f_{22} & +\,0 + f_{22} & +\,f_{21} - 0 \\
f_{21} & f_{21} + 0 & 0 - f_{22} \\
-\; f_{12} & -\,0 - f_{12} & -\,f_{11} + 0.
\end{array}$$

In ordinary notation this system is

$$\begin{array}{c|cc}
 & f_1 & f_2 \\
\hline
f_1 & f_1 & f_2 \\
f_2 & f_2 & -f_1.
\end{array}$$

So the given system is the compound of the Cayley two-unit system and the ordinary complex system.

III. Let us take up next the system whose multiplication table is

	h_1	h_2	h_3	h_4	h_5	h_6
h_1	h_1	h_2	h_3	h_4	h_5	h_6
h_2	h_2	0	h_4	0	h_6	0
h_3	h_3	h_4	0	0	0	0
h_4	h_4	0	0	0	0	0
h_5	h_5	h_6	0	0	0	0
h_6	h_6	0	0	0	0	0.

If this is a composite system, it may be either the product of a two-unit system by a three-unit system or the product of a three-unit system by a two-unit system.* Assume for them

	e_1	e_2
e_1	e_{11}	e_{12}
e_2	e_{21}	e_{22}

and

	f_1	f_2	f_3
f_1	f_{11}	f_{12}	f_{13}
f_2	f_{21}	f_{22}	f_{23}
f_3	f_{31}	f_{32}	f_{33}.

Symbolically the compound system has two possible forms

	e_1f_1	e_1f_2	e_1f_3	e_2f_1	e_2f_2	e_2f_3
e_1f_1	$e_{11}f_{11}=g_{11}$	$e_{11}f_{12}=g_{12}$	$e_{11}f_{13}=g_{13}$	$e_{12}f_{11}=g_{14}$	$e_{12}f_{12}=g_{15}$	$e_{12}f_{13}=g_{16}$
e_1f_2	$e_{11}f_{21}=g_{21}$	$e_{11}f_{22}=g_{22}$	$e_{11}f_{23}=g_{23}$	$e_{12}f_{21}=g_{24}$	$e_{12}f_{22}=g_{25}$	$e_{12}f_{23}=g_{26}$
e_1f_3	$e_{11}f_{31}=g_{31}$	$e_{11}f_{32}=g_{32}$	$e_{11}f_{33}=g_{33}$	$e_{12}f_{31}=g_{34}$	$e_{12}f_{32}=g_{35}$	$e_{12}f_{33}=g_{36}$
e_2f_1	$e_{21}f_{11}=g_{41}$	$e_{21}f_{12}=g_{42}$	$e_{21}f_{13}=g_{43}$	$e_{22}f_{11}=g_{44}$	$e_{22}f_{12}=g_{45}$	$e_{22}f_{13}=g_{46}$
e_2f_2	$e_{21}f_{21}=g_{51}$	$e_{21}f_{22}=g_{52}$	$e_{21}f_{23}=g_{53}$	$e_{22}f_{21}=g_{54}$	$e_{22}f_{22}=g_{55}$	$e_{22}f_{23}=g_{56}$
e_2f_3	$e_{21}f_{31}=g_{61}$	$e_{21}f_{32}=g_{62}$	$e_{21}f_{33}=g_{63}$	$e_{22}f_{31}=g_{64}$	$e_{22}f_{32}=g_{65}$	$e_{22}f_{33}=g_{66}$

(12)

and

	e_1f_1	e_2f_1	e_1f_2	e_2f_2	e_1f_3	e_2f_3
e_1f_1	$e_{11}f_{11}=g_{11}$	$e_{12}f_{11}=g_{12}$	$e_{11}f_{12}=g_{13}$	$e_{12}f_{12}=g_{14}$	$e_{11}f_{13}=g_{15}$	$e_{12}f_{13}=g_{16}$
e_2f_1	$e_{21}f_{11}=g_{21}$	$e_{22}f_{11}=g_{22}$	$e_{21}f_{12}=g_{23}$	$e_{22}f_{12}=g_{24}$	$e_{21}f_{13}=g_{25}$	$e_{22}f_{13}=g_{26}$
e_1f_2	$e_{11}f_{21}=g_{31}$	$e_{12}f_{21}=g_{32}$	$e_{11}f_{22}=g_{33}$	$e_{12}f_{22}=g_{34}$	$e_{11}f_{23}=g_{35}$	$e_{12}f_{23}=g_{36}$
e_2f_2	$e_{21}f_{21}=g_{41}$	$e_{22}f_{21}=g_{42}$	$e_{21}f_{22}=g_{43}$	$e_{22}f_{22}=g_{44}$	$e_{21}f_{23}=g_{45}$	$e_{22}f_{23}=g_{46}$
e_1f_3	$e_{11}f_{31}=g_{51}$	$e_{12}f_{31}=g_{52}$	$e_{11}f_{32}=g_{53}$	$e_{12}f_{32}=g_{54}$	$e_{11}f_{33}=g_{55}$	$e_{12}f_{33}=g_{56}$
e_2f_3	$e_{21}f_{31}=g_{61}$	$e_{22}f_{31}=g_{62}$	$e_{21}f_{32}=g_{63}$	$e_{22}f_{32}=g_{64}$	$e_{21}f_{33}=g_{65}$	$e_{22}f_{33}=g_{66}$.

(13)

* The systems resulting from the two orders of combination are essentially the same. The uncertainty is one of subscripts in the identification with the symbolic products and may be encountered in factoring any composite system having factors of unequal orders. However the number of trials that may be found necessary is always finite.

The multiplication table of the given system determines the γ's. The substitution of their values in (10) gives

$$
\begin{aligned}
g_1' &= g_{11} + g_{22} + g_{33} + g_{44} + g_{55} + g_{66} \\
g_2' &= g_{21} + g_{43} + g_{65} \\
g_3' &= g_{31} + g_{42} \\
g_4' &= g_{41} \\
g_5' &= g_{51} + g_{62} \\
g_6' &= g_{61} .
\end{aligned}
$$

Upon substitution of the symbolic products (12) one obtains

$$
\begin{aligned}
g_1' &= e_{11}f_{11} + e_{11}f_{22} + e_{11}f_{33} + e_{22}f_{11} + e_{22}f_{22} + e_{22}f_{33} \\
&= (e_{11} + e_{22})(f_{11} + f_{22} + f_{33}) \\
g_2' &= e_{11}f_{21} + e_{21}f_{13} + e_{22}f_{32} \\
g_3' &= e_{11}f_{31} + e_{21}f_{12} \\
g_4' &= e_{21}f_{11} \\
g_5' &= e_{21}f_{21} + e_{21}f_{32} = e_{21}(f_{21} + f_{32}) \\
g_6' &= e_{21}f_{31} .
\end{aligned}
$$

The units g_2' and g_3' do not factor and therefore the second order of combination (13) must be tried:

$$
\begin{aligned}
g_1' &= e_{11}f_{11} + e_{22}f_{11} + e_{11}f_{22} + e_{22}f_{22} + e_{11}f_{33} + e_{22}f_{33} \\
&= (e_{11} + e_{22})(f_{11} + f_{22} + f_{33}) \\
g_2' &= e_{21}f_{11} + e_{21}f_{22} + e_{21}f_{33} = e_{21}(f_{11} + f_{22} + f_{33}) \\
g_3' &= e_{11}f_{21} + e_{22}f_{21} = (e_{11} + e_{22})f_{21} \\
g_4' &= e_{21}f_{21} \qquad\qquad = e_{21}(f_{21}) \\
g_5' &= e_{11}f_{31} + e_{22}f_{31} = (e_{11} + e_{22})f_{31} \\
g_6' &= e_{21}f_{31} \qquad\qquad = e_{21}(f_{31}).
\end{aligned}
$$

This time factors appear and the units of one system are represented by $e_{11} + e_{22}$ and e_{21}. Multiplying out according to the law given above, we obtain for the first factor system

$$
\begin{array}{c|cc}
 & e_{11} + e_{22} & e_{21} \\
\hline
e_{11} + & e_{11} + 0 & 0 \\
\quad e_{22} & + 0 + e_{22} & + e_{21} \\
e_{21} & e_{21} + 0 & 0.
\end{array}
$$

In ordinary notation this system is

$$
\begin{array}{c|cc}
 & e_1 & e_2 \\
\hline
e_1 & e_1 & e_2 \\
e_2 & e_2 & 0.
\end{array}
$$

The units of the second factor system are represented by $f_{11}+f_{22}+f_{33}, f_{21}$, and f_{31}. These determine the system

$$
\begin{array}{r|ccc}
 & f_{11}+f_{22}+f_{33} & f_{21} & f_{31} \\
\hline
f_{11} & f_{11}+0+0 & 0 & 0 \\
+f_{22} & +0+f_{22}+0 & +f_{21} & +0 \\
+f_{33} & +0+0+f_{33} & +0 & +f_{31} \\
f_{21} & f_{21}+0+0 & 0 & 0 \\
f_{31} & f_{31}+0+0 & 0 & 0.
\end{array}
$$

In ordinary notation this system is

$$
\begin{array}{c|ccc}
 & f_1 & f_2 & f_3 \\
\hline
f_1 & f_1 & f_2 & f_3 \\
f_2 & f_2 & 0 & 0 \\
f_3 & f_3 & 0 & 0.
\end{array}
$$

IV. The factoring of a composite system of eight units into a two-unit system and a four-unit system* presents no new difficulties and the details of the method may be readily developed from the two preceding examples. By this scheme the octonian system is easily shown to be the compound of the ordinary complex system and the quaternion system.

V. This method at times furnishes curious results. To exhibit this, let us apply the method to the system†

$$
\begin{array}{c|cccc}
 & h_1 & h_2 & h_3 & h_4 \\
\hline
h_1 & h_1 & h_2 & h_3 & h_4 \\
h_2 & h_2 & 0 & h_4 & 0 \\
h_3 & h_3 & -h_4 & 0 & 0 \\
h_4 & h_4 & 0 & 0 & 0.
\end{array}
$$

Writing out (10) and substituting from (11)

$$
\begin{aligned}
g'_1 &= g_{11}+g_{22}+g_{33}+g_{44} = e_{11}f_{11}+e_{22}f_{11}+e_{11}f_{22}+e_{22}f_{22} \\
&= (e_{11}+e_{22})\,(f_{11}+f_{22}) \\
g'_2 &= g_{21}+g_{43} = e_{21}f_{11}+e_{21}f_{22} = e_{21}\,(f_{11}+f_{22}) \\
g'_3 &= g_{31}-g_{42} = e_{11}f_{21}-e_{22}f_{21} = (e_{11}-e_{22})f_{21} \\
g'_4 &= g_{41} \qquad\quad = e_{21}f_{21} \qquad\qquad = e_{21}\,(f_{21}).
\end{aligned}
$$

*Of course the four-unit system itself may be factorable.

†Study, Encyklopaedie der Math. Wissen., vol. 1, p. 167 system VIII.

Here the factors show two units in one system, $f_{11}+f_{22}$ and f_{21}, and for the other system three independent units, $e_{11}+e_{22}$, e_{21}, and $e_{11}-e_{22}$. The corresponding systems are

	$f_{11}+f_{22}$	f_{21}
$f_{11}+f_{22}$	$f_{11}+f_{22}$	f_{21}
f_{21}	f_{21}	0

and

	$e_{11}+e_{22}$	e_{21}	$e_{11}-e_{22}$
$e_{11}+e_{22}$	$e_{11}+e_{22}$	e_{21}	$e_{11}-e_{22}$
e_{21}	e_{21}	0	e_{21}
$e_{11}-e_{22}$	$e_{11}-e_{22}$	$-e_{21}$	$e_{11}+e_{22}$.

In ordinary notation these systems are

	f_1	f_2
f_1	f_1	f_2
f_2	f_2	0

and *

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	0	e_2
e_3	e_3	$-e_2$	e_1.

The compound system is

	$e_1 f_1$	$e_2 f_1$	$e_3 f_2$	$e_2 f_2$	$e_3 f_1$	$e_1 f_2$
$e_1 f_1$	$e_1 f_1$	$e_2 f_1$	$e_3 f_2$	$e_2 f_2$	$e_3 f_1$	$e_1 f_2$
$e_2 f_1$	$e_2 f_1$	$0 . f_1$	$e_2 f_2$	$0 . f_2$	$e_2 f_1$	$e_2 f_2$
$e_3 f_2$	$e_3 f_2$	$-e_2 f_2$	$e_1 . 0$	$-e_2 . 0$	$e_1 f_2$	$e_3 . 0$
$e_2 f_2$	$e_2 f_2$	$0 . f_2$	$e_2 . 0$	$0 . 0$	$e_2 f_2$	$e_2 . 0$
$e_3 f_1$	$e_3 f_1$	$-e_2 f_1$	$e_1 f_2$	$-e_2 f_2$	$e_1 f_1$	$e_3 f_2$
$e_1 f_2$	$e_1 f_2$	$e_2 f_2$	$e_3 . 0$	$e_2 . 0$	$e_3 f_2$	$e_1 . 0$

or

	h_1	h_2	h_3	h_4	h_5	h_6
h_1	h_1	h_2	h_3	h_4	h_5	h_6
h_2	h_2	0	h_4	0	h_2	h_4
h_3	h_3	$-h_4$	0	0	h_6	0
h_4	h_4	0	0	0	h_4	0
h_5	h_5	$-h_2$	h_6	$-h_4$	h_1	h_3
h_6	h_6	h_4	0	0	h_3	0.

Our given system appears as a sub-system of the six-unit system.

---

* By a change in the order of units, the system

	e_1	e_3	e_2
e_1	e_1	e_3	e_2
e_3	e_3	e_1	$-e_2$
e_2	e_2	e_2	0

is seen to be the reciprocal of system (33) II. Encyk. der Math. Wissen., vol. 1, p. 167.

Next consider the system *

	h_1	h_2	h_3	h_4
h_1	h_1	h_2	h_3	h_4
h_2	h_2	$-h_1$	h_4	$-h_3$
h_3	h_3	$-h_4$	0	0
h_4	h_4	h_3	0	0.

Writing out (10) and substituting from (11)

$$\begin{aligned}
g_1' &= g_{11} + g_{22} + g_{33} + g_{44} = e_{11}f_{11} + e_{22}f_{11} + e_{11}f_{22} + e_{22}f_{22} \\
&= (e_{11} + e_{22})\,(f_{11} + f_{22}) \\
g_2' &= g_{21} - g_{12} + g_{43} - g_{34} = e_{21}f_{11} - e_{12}f_{11} + e_{21}f_{22} - e_{12}f_{22} \\
&= (e_{21} - e_{12})\,(f_{11} + f_{22}) \\
g_3' &= g_{31} - g_{42} = e_{11}f_{21} - e_{22}f_{21} = (e_{11} - e_{22})\,f_{21}, \\
g_4' &= g_{41} + g_{32} = e_{21}f_{21} + e_{12}f_{21} = (e_{21} + e_{12})\,f_{21}.
\end{aligned}$$

The factors show four independent expressions for the units of one system and two for the units of the other system. For the first, the table is

	$e_{11}+e_{22}$	$e_{21}-e_{12}$	$e_{11}-e_{22}$	$e_{21}+e_{12}$
$e_{11}+e_{22}$	$e_{11}+e_{22}$	$e_{21}-e_{12}$	$e_{11}-e_{22}$	$e_{21}+e_{12}$
$e_{21}-e_{12}$	$e_{21}-e_{12}$	$-e_{11}-e_{22}$	$e_{21}+e_{12}$	$-e_{11}+e_{22}$
$e_{11}-e_{22}$	$e_{11}-e_{22}$	$-e_{12}-e_{21}$	$e_{11}+e_{22}$	$e_{12}-e_{21}$
$e_{21}+e_{12}$	$e_{21}+e_{12}$	$e_{11}-e_{22}$	$e_{21}-e_{12}$	$e_{11}+e_{22}$.

In ordinary notation these two systems are

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$-e_4$	e_1	$-e_2$
e_4	e_4	e_3	e_2	e_1

and

	f_1	f_2
f_1	f_1	f_2
f_2	f_2	0.

* Encyk., vol. 1, p. 167 system VII *a*.

The compound system is

	e_1f_1	e_2f_1	$e_3\ f_2$	$e_4\ f_2$	e_4f_1	e_3f_1	$e_2\ f_2$	$e_1\ f_2$
e_1f_1	e_1f_1	e_2f_1	$e_3\ f_2$	$e_4\ f_2$	e_4f_1	e_3f_1	$e_2\ f_2$	$e_1\ f_2$
e_2f_1	e_2f_1	$-e_1f_1$	$e_4\ f_2$	$-e_3\ f_2$	$-e_3f_1$	e_4f_1	$-e_1\ f_2$	$e_2\ f_2$
e_3f_2	e_3f_2	$-e_4f_2$	$e_1 \cdot 0$	$-e_2 \cdot 0$	$-e_2f_2$	e_1f_2	$-e_4 \cdot 0$	$e_3 \cdot 0$
e_4f_2	e_4f_2	e_3f_2	$e_2 \cdot 0$	$e_1 \cdot 0$	e_1f_2	e_2f_2	$e_3 \cdot 0$	$e_4 \cdot 0$
e_4f_1	e_4f_1	e_3f_1	$e_2\ f_2$	$e_1\ f_2$	e_1f_1	e_2f_1	$e_3\ f_2$	$e_4\ f_2$
e_3f_1	e_3f_1	$-e_4f_1$	$e_1\ f_2$	$-e_2\ f_2$	$-e_2f_1$	e_1f_1	$-e_4\ f_2$	$e_3\ f_2$
e_2f_2	e_2f_2	$-e_1f_2$	$e_4 \cdot 0$	$-e_3 \cdot 0$	$-e_3f_2$	e_4f_2	$-e_1 \cdot 0$	$e_2 \cdot 0$
e_1f_2	e_1f_2	e_2f_2	$e_3 \cdot 0$	$e_4 \cdot 0$	e_4f_2	e_3f_2	$e_2 \cdot 0$	$e_1 \cdot 0$

or

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e_1$	e_4	$-e_3$	$-e_6$	e_5	$-e_8$	e_7
e_3	e_3	$-e_4$	0	0	$-e_7$	e_8	0	0
e_4	e_4	e_3	0	0	e_8	e_7	0	0
e_5	e_5	e_6	e_7	e_8	e_1	e_2	e_3	e_4
e_6	e_6	$-e_5$	e_8	$-e_7$	$-e_2$	e_1	$-e_4$	e_3
e_7	e_7	$-e_8$	0	0	$-e_3$	e_4	0	0
e_8	e_8	e_7	0	0	e_4	e_3	0	0.

Our given system is a sub-system of the eight-unit system. This system is peculiar. Let $j = 1, 2, 3, 4$ and $k = 5, 6, 7, 8$, then

$$e_{j_1} e_{j_2} = e_{j_3}, \quad e_{j_1} e_{k_2} = e_{k_3}, \quad e_{k_1} e_{j_2} = e_{k_3}, \text{ and } e_{k_1} e_{k_2} = e_{j_3}.$$

§5.—DIVISORS OF ZERO.

The product of $x = \sum_{i_1} x_{i_1} e_{i_1}$ and $y = \sum_{i_2} y_{i_2} e_{i_2}$ in the system E is

$$xy = \sum_{i_1 i_2 i_3} x_{i_1} y_{i_2} \gamma_{i_1 i_2 i_3} e_{i_3} = \sum_{i_3} z_{i_3} e_{i_3}. \tag{14}$$

If

$$\Delta_x \equiv \left| \begin{matrix} \sum_{i_1} x_{i_1} \gamma_{i_1 i_2 i_3} \\ i_2, i_3 = 1, \ldots, n \end{matrix} \right| \equiv 0, \tag{15}$$

then the number x is called a left-hand divisor of zero. Similarly if

$$\Delta'_y \equiv \left| \begin{matrix} \sum_{i_2} y_{i_2} \gamma_{i_1 i_2 i_3} \\ i_1, i_3 = 1, \ldots, n \end{matrix} \right| \equiv 0, \tag{16}$$

then the number y is called a right-hand divisor of zero. The substitution of $\mu = 0$ in (4) gives a form which is evidently (15) and consequently Δ_x is the absolute term in that type of characteristic equation. Similarly Δ'_y is the absolute term in the other type of characteristic equation suggested in a previous foot note (§2).

If the absolute term of the characteristic equation of the general number of a system vanishes, then every number of that system is a left-hand divisor of zero. It is known that in every system except the real, the ordinary complex and the quaternion, special numbers can be found for which the characteristic equation has no absolute term and such numbers are divisors of zero.

From the theory of equations it is plain that* ${}_E\Delta_x$ is the product of the n roots, μ_i, of the characteristic equation of a number of the system E and that ${}_F\Delta_x$ is the product of the r roots, ν_j, of the characteristic equation of a number of the system F. Then the absolute term of the characteristic equation of their compound number is the product of the nr roots, $\mu_i \nu_j$, and in this product each root μ_i occurs r times and each root ν_j occurs n times. Therefore

$${}_{EF}\Delta_x = ({}_E\Delta_x)^r \cdot ({}_F\Delta_x)^n. \tag{17}$$

From (17) it is evident that if either of the factor numbers is a divisor of zero, then the compound number must be a divisor of zero.

If the absolute term of the characteristic equation of a general composite number of a composite system vanishes, then every composite number of this system is a left-hand divisor of zero and in the factor systems every number of one (or perhaps of both) is a left-hand divisor of zero.

If the general composite number is not a divisor of zero, it may still be that there are special composite numbers which are divisors of zero (that is, while ${}_{EF}\Delta_x$ does not vanish identically, it may vanish for special values of the x's). In this case, it follows as above that at least one of the factors of the composite number is a divisor of zero.

THE UNIVERSITY OF COLORADO, *June*, 1906.

---

* The first subscript indicates the system under consideration.